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Majorizing sequences for iterative methods

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ABSTRACT

We provide convergence results for very general majorizing sequences of iterative methods. Using our new concept of recurrent functions, we unify the semilocal convergence analysis of Newton-type methods (NTM) under more general Lipschitz-type conditions. We present two very general majorizing sequences and we extend the applicability of (NTM) using the same information before Chen and Yamamoto (1989) [13], Deuflhard (2004) [16], Kantorovich and Akilov (1982) [19], Miel (1979) [20], Miel (1980) [21] and Rheinboldt (1968) [30]. Applications, special cases and examples are also provided in this study to justify the theoretical results of our new approach.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) + G(x) = 0, \tag{1.1}$$

where, F is a Fréchet-differentiable operator defined on a open subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in $\mathcal{X}$ and $G : \mathcal{D} \longrightarrow \mathcal{X}$ is a continuous operator.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = T(x)$, for some suitable operator T, where x is the state. Then the equilibrium states are determined by solving Eq. (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative—when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such

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cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

Newton-type methods (NTM)

$$x_{n+1} = x_n - A_n^{-1}(F(x_n) + G(x_n)) \quad (n \geq 0),\ (x_0 \in \mathcal{D}), \tag{1.2}$$

are used to generate a sequence $\{x_n\}$ approximating $x^\star$ [1–33]. Here, $A(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$ is an approximation to Fréchet-derivative F [5]. The most popular choice for operators A and G is $A(x) = F'(x)$ and $G(x) = 0,\ (x \in \mathcal{D})$. Then, we obtain the Newton method (NM)

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad (x_0 \in \mathcal{D}),\ (n \geq 0). \tag{1.3}$$

Another interesting choice of operator A is

$$A(x) = \mathcal{I} - \frac{1}{2}F'(x)^{-1}F''(x)F'(x)^{-1}F(x),$$

then, if $G(x) = 0$ $(x \in \mathcal{D})$, (1.2) becomes Halley's method

$$x_{n+1} = x_n - \left(\mathcal{I} - \frac{1}{2}F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n)\right)^{-1} F(x_n), \quad (x_0 \in \mathcal{D}),\ (n \geq 0). \tag{1.4}$$

Many other choices for A and G are possible [1,5,8,12]. A local as well as a semilocal convergence analysis for (NTM) have been given by many authors under various Lipschitz-type conditions [1–4,6,11–33]. A survey of recent such results can be found in [4,8] (see also the references there).

Here, we unify the semilocal study of iterative methods under more general Lipschitz-type conditions and majorizing sequences than before [13,16,19–21,30]. In particular, we show that it is possible to expand the applicability of iterative methods and improve the error bound involved using the same information as before.

Majorizing sequences play important role in the study of iterative methods. We define a very general majorizing sequence. Let $\eta \geq 0$ and $\alpha_n \geq 0$ be given. Define scalar sequence $\{t_n\}$ $(n \geq 0)$ by

$$t_0 = 0, \qquad t_1 = \eta, \qquad t_{n+2} = t_{n+1} + \alpha_n(t_{n+1} - t_n). \tag{1.5}$$

If there exists α such that

$$\alpha_n \leq \alpha < 1 \quad (n \geq 1), \tag{1.6}$$

then, sequence $\{t_n\}$ $(n \geq 0)$, is well defined, non-decreasing, bounded above by:

$$t^{\star\star} = \frac{\eta}{1 - \alpha} \tag{1.7}$$

and converges to its unique least upper bound $t^\star$ satisfying

$$0 \leq t^\star \leq t^{\star\star}. \tag{1.8}$$

Moreover the following estimates hold for all $n \geq 1$:

$$0 \leq t_{n+2} - t_{n+1} \leq \alpha(t_{n+1} - t_n) \leq \alpha^{n+1}\eta \tag{1.9}$$

and

$$t^\star - t_n \leq \frac{\eta}{1 - \alpha}\alpha^n \leq t^{\star\star}. \tag{1.10}$$

For example, many majorizing sequences of (NTM) (1.2) are special cases of (1.5) [5]. Therefore, choosing sequence $\{\alpha_n\}$ and also verifying (1.6) is very important.

In this study, we provide conditions implying (1.6), when for $\beta_n \geq 0$, $\gamma_n \geq 0$ $(n \geq 0)$:

$$\alpha_n = \frac{\beta_n}{1 - \gamma_n t_{n+1}}. \tag{1.11}$$

Note that (1.6) is obviously such a condition.

Sequence (1.5) for α_n given by (1.11) becomes

$$t_0 = 0, \qquad t_1 = \eta, \qquad t_{n+2} = t_{n+1} + \frac{\beta_n(t_{n+1} - t_n)}{1 - \gamma_n t_{n+1}}. \tag{1.12}$$

We also examine the special case of (1.11), when for $\delta_n \geq 0$:

$$\beta_n = \delta_n(t_{n+1} - t_n). \tag{1.13}$$

In this case (1.12) can be written as

$$t_0 = 0, \qquad t_1 = \eta, \qquad t_{n+2} = t_{n+1} + \frac{\delta_n(t_{n+1} - t_n)^2}{1 - \gamma_n t_{n+1}}. \tag{1.14}$$

The paper in organized as follows: Sections 2 and 3 contain convergence results of majorizing sequences (1.12) and (1.14), respectively. The semilocal convergence result for (NTM) is given in Section 4. Finally, applications, numerical examples and special cases are provided in the concluding Section 5.

2. Convergence of majorizing sequence (1.12)

We can show the following convergence result for (1.12).

Lemma 2.1. *Let* $\eta > 0$, $\gamma > 0$, $\{\beta_n\} > 0$ *and* $\gamma_n > 0$. *Assume that together with*

$$\gamma_0 \eta < 1 \tag{2.1}$$

and

$$\gamma_n \leq \gamma, \tag{2.2}$$

one of the set of hypotheses holds

(I) *There exists parameter* α *such that*

$$\lim_{n \longrightarrow \infty} \beta_n = 0, \tag{2.3}$$

$$g_n(\alpha) = \beta_{n+1} - \beta_n + \alpha \left(\gamma_{n+1} \frac{1-\alpha^{n+2}}{1-\alpha} - \gamma_n \frac{1-\alpha^{n+1}}{1-\alpha} \right) \eta \geq 0 \tag{2.4}$$

and

$$\frac{\beta_0}{1-\gamma_0 \eta} \leq \alpha \leq 1 - \gamma \eta. \tag{2.5}$$

(II) *There exists parameter* α *such that*

$$g_n(\alpha) \leq 0, \tag{2.6}$$

$$f_1(\alpha) \leq 0 \tag{2.7}$$

and

$$\frac{\beta_0}{1-\gamma_0 \eta} \leq \alpha \leq 1 - \gamma \eta,$$

where,

$$f_1(s) = \beta_1 + s\gamma_1(1+s)\eta - s. \tag{2.8}$$

Then scalar sequence defined by (1.5) *satisfies* (1.7)–(1.10).

Proof.

Case (I). We shall use induction to show (1.9). We must have

$$0 \leq \frac{\beta_n}{1-\gamma_n t_{n+1}} \leq \alpha. \tag{2.9}$$

Estimate (2.9) holds for $n = 0$ by (2.5). Then, we have by (1.12) (for $n = 0$) that

$$t_2 - t_1 \leq \alpha(t_1 - t_0) = \alpha\eta \quad \text{and} \quad t_2 \leq \eta + \alpha\eta = \frac{1-\alpha^2}{1-\alpha}\eta.$$

Assume (2.9) holds for all $k \leq n$. Then, we have again by (1.12) that

$$t_{k+1} - t_k \leq \alpha(t_k - t_{k-1}) \leq \alpha^k \eta$$

and

$$t_{k+1} \leq t_k + \alpha^k \eta \leq t_{k-1} + (\alpha^{k-1} + \alpha^k)\eta \leq t_1 + (\alpha + \cdots + \alpha^{k-1} + \alpha^k)\eta = \frac{1-\alpha^{k+1}}{1-\alpha}\eta.$$

Using this expression for t_{k+1}, we can show instead of (2.9) that

$$\beta_n + \alpha\gamma_n \frac{1-\alpha^{n+1}}{1-\alpha}\eta - \alpha \leq 0. \tag{2.10}$$

Estimate (2.10) motivates us to introduce recurrent functions f_n on $[0, \infty)$ by

$$f_n(s) = \beta_n + s\gamma_n(1 + s + \cdots + s^n)\eta - s \leq 0. \tag{2.11}$$

We need a relationship between two consecutive recurrent functions f_n:

$$f_{n+1}(s) = \beta_{n+1} + s\gamma_{n+1}(1 + s + \cdots + s^{n+1})\eta - s - \beta_n - s\gamma_n(1 + s + \cdots + s^n)\eta + s + f_n(s),$$

so,

$$f_{n+1}(s) = f_n(s) + \beta_{n+1} - \beta_n + s(\gamma_{n+1}(1 + s + \cdots + s^{n+1}) - \gamma_n(1 + s + \cdots + s^n))\eta. \tag{2.12}$$

Estimate (2.10) certainly holds, if

$$f_n(\alpha) \leq 0. \tag{2.13}$$

Using hypotheses (2.4) and (2.12) we have:

$$f_n(\alpha) \leq f_{n+1}(\alpha). \tag{2.14}$$

Define function f_∞ on $[0, 1)$ by

$$f_\infty(s) = \lim_{n \longrightarrow \infty} f_n(s). \tag{2.15}$$

In view of (2.2), (2.11) and (2.15), we get

$$f_\infty(\alpha) \leq \left(\frac{\gamma\eta}{1 - \alpha} - 1\right)\alpha \tag{2.16}$$

and

$$f_n(\alpha) \leq f_\infty(\alpha). \tag{2.17}$$

Hence, (2.13) is satisfied, if

$$\frac{\gamma\eta}{1 - \alpha} - 1 \leq 0, \tag{2.18}$$

which is true by (2.5).

The induction for (1.9) is completed.

Case (II). We must again show (2.13). This time by (2.6), (2.7) and (2.12), we have

$$f_2(\alpha) = f_1(\alpha) + g_1(\alpha) \leq 0. \tag{2.19}$$

Assume $f_k(\alpha) \leq 0$ for all $k \leq n$. Then, again by (2.6) and (2.12), we get

$$f_{n+1}(\alpha) = f_n(\alpha) + g_n(\alpha) \leq 0. \tag{2.20}$$

We also have by (2.13) and (2.15)

$$f_\infty(\alpha) = \lim_{n \longrightarrow \infty} f_n(\alpha) \leq 0.$$

That completes the induction.

Therefore, sequence $\{t_n\}$ is non-decreasing, bounded above by $t^{\star\star}$ and as such it converges to its unique least bound $t^\star$ satisfying (1.8).

Finally, estimate (1.10) follows from (1.9) by using standard majorization techniques [5,8,19].

That completes the proof of Lemma 2.1. □

3. Convergence of majorizing sequence (1.14)

Lemma 3.1. *Let $\gamma_n > 0$, $\delta_n > 0$ and $\eta > 0$ be given parameters. Define functions $\{f_n\}$ $(n \geq 1)$, $\{g_n\}$ $(n \geq 0)$ on $[0, \infty)$ by*

$$f_n(s) = \delta_n s^{n-1}\eta + \gamma_n(1 + s + \cdots + s^n)\eta - 1 \tag{3.1}$$

and

$$g_n(s) = \gamma_{n+1}s^{n+1} + (\delta_{n+1} + \gamma_{n+1} - \gamma_n)s^n + (\gamma_{n+1} - \gamma_n - \delta_n)s^{n-1} + (\gamma_{n+1} - \gamma_n)s^{n-2} + \cdots + (\gamma_{n+1} - \gamma_n). \tag{3.2}$$

Assume there exist parameters α, γ such that

$$\gamma_0\eta < 1, \qquad \gamma_n \leq \gamma \tag{3.3}$$

and

$$\frac{\delta_0}{1 - \gamma_0\eta} \leq \alpha \leq 1 - \gamma\eta. \tag{3.4}$$

We also suppose that the following assumptions (I) *or* (II) *hold:*

(I) *There exists parameter* α *such that*

$$g_n(\alpha) \geq 0 \tag{3.5}$$

and

$$f_\infty(\alpha) \leq 0, \tag{3.6}$$

where,

$$f_\infty(s) = \lim_{n \longrightarrow \infty} f_n(s), \tag{3.7}$$

(II)

$$f_1(\alpha) \leq 0 \tag{3.8}$$

and

$$g_n(\alpha) \leq 0. \tag{3.9}$$

Then scalar sequence defined by (1.14) *is well defined, non-decreasing, bounded from above by:*

$$t^{\star\star} = \frac{\eta}{1-\alpha} \tag{3.10}$$

and converges to its unique least upper bound $t^\star$ *satisfying*

$$0 \leq t^\star \leq t^{\star\star}. \tag{3.11}$$

Moreover, the following estimates hold for all $n \geq 1$*:*

$$0 \leq t_{n+2} - t_{n+1} \leq \alpha(t_{n+1} - t_n) \leq \alpha^{n+1}\eta \tag{3.12}$$

and

$$0 \leq t^\star - t_n \leq \frac{\eta}{1-\alpha}\alpha^n \leq t^{\star\star}. \tag{3.13}$$

Proof. We shall show using induction that

$$0 < \frac{\delta_n(t_{n+1} - t_n)}{1 - \gamma_n t_{n+1}} \leq \alpha \quad (n \geq 0). \tag{3.14}$$

Estimate (3.12) will then follow from (1.14) and (3.14). Inequality (3.14) holds for $n = 0$ by (3.4) and (1.14). That is (3.12) holds for $n = 0$. Assume (3.14) holds for all $k \leq n$.

Replacing the expression of t_k in (2.9), we obtain that estimate (3.14) certainly holds if

$$\delta_n \alpha^k \eta + \alpha \gamma_k \frac{1-\alpha^{k+1}}{1-\alpha}\eta - \alpha \leq 0, \tag{3.15}$$

or

$$\delta_n \alpha^{k-1}\eta + \gamma_k(1 + \alpha + \cdots + \alpha^k)\eta - 1 \leq 0 \leq 0. \tag{3.16}$$

Estimate (3.16) motivates us to define recurrent functions f_k $(k \geq 0)$ given by (3.1). Estimate (3.14) is true, if

$$f_k(\alpha) \leq 0 \quad (k \geq 1). \tag{3.17}$$

We need a relationship between two consecutive functions f_k:

$$f_{k+1}(s) = \delta_{k+1}s^k\eta + \gamma_{k+1}(1 + s + \cdots + s^{k+1})\eta - 1 - \delta_k s^{k-1}\eta - \gamma_k(1 + s + \cdots + s^k)\eta + 1 + f_k(s)$$

so,

$$f_{k+1}(s) = f_k(s) + g_k(s)\eta, \tag{3.18}$$

where, g_k is given by (3.2).

Case of (I). It then follows from (3.5), (3.7) and (3.18) that

$$f_\infty(s) \geq f_{k+1}(\alpha) \geq f_k(\alpha) \quad \text{for all } k. \tag{3.19}$$

Hence, (3.17) holds by (3.6) and (3.19).

Case of (II). Using (3.8), (3.9) and (3.18), we get

$$f_2(\alpha) = f_1(\alpha) + g_1(\alpha)\eta \leq 0. \tag{3.20}$$

That is (3.17) holds for $k = 1$.

Assume $f_k(\alpha) \leq 0$ for all $k \leq n$, then again by (3.9) and (3.18)

$$f_{k+1}(\alpha) = f_k(\alpha) + g_k(\alpha)\eta \leq 0,$$

which shows (3.18) for all k.
Note that

$$f_\infty(\alpha) = \lim_{n\longrightarrow\infty} f_n(\alpha) \leq 0.$$

The induction for (3.18) is completed.

Hence, sequence $\{t_n\}$ is non-decreasing, bounded above by $t^{\star\star}$ and as such it converges to $t^\star$ satisfying (3.9). Finally, estimate (3.13) follows from (2.13) by using standard majorization techniques [5,19].

That completes the proof of Lemma 3.1. □

We need the following assumptions:

$(\mathcal{C}_1)$ There exist a, b, α and unique positive zero denoted by s_n for each function g_n such that

$$\gamma_n \leq a,$$
$$\delta_n \leq b$$

and

$$\max\left\{\frac{\delta_0\eta}{1-\gamma_0\eta}, s^\star\right\} \leq \alpha \leq 1 - a\eta,$$

where, $s^\star = \sup_{n\geq 0} s_n$.

$(\mathcal{C}_2)$ There exist a, b, α such that

$$\gamma_{n+1} < \gamma_n \leq a,$$
$$\delta_n \leq b$$

and

$$\max\left\{\frac{\delta_0\eta}{1-\gamma_0\eta}, s^\star\right\} \leq \alpha \leq 1 - a\eta.$$

Note that by (3.2) and $\gamma_{n+1} < \gamma_n$, we have $g_n(0) = \gamma_{n+1} - \gamma_n < 0$. Moreover $\lim_{s\longrightarrow\infty} g_n(s) = \infty$. It follows by the intermediate value theorem that each function g_n has at least one positive zero s_n. By Descarte's rule of signs, s_n is the unique positive zero of g_n.

$(\mathcal{C}_3)$ There exist a, b, α such that

$$\gamma_n \leq \gamma_{n+1} \leq a,$$
$$\delta_n \leq b$$

and

$$\max\left\{\frac{\delta_0\eta}{1-\gamma_0\eta}, r^\star\right\} \leq \alpha \leq 1 - a\eta,$$

where,

$$r_n = \frac{2\delta_n}{\delta_{n+1} + \sqrt{\delta_{n+1}^2 + 4\gamma_{n+1}\delta_n}} \quad \text{and} \quad r^\star = \sup r_n.$$

$(\mathcal{C}_{11})$ Each function g_n has a minimal zero s_n in (0, 1) and there exists α such that

$$\frac{\delta_0\eta}{1-\gamma_0\eta} \leq \alpha \leq \min\{s^\star, s_0\} < 1,$$

where,

$$s^\star = \inf s_n \quad \text{and} \quad s_0 = \frac{1-(\gamma_1+\delta_1)\eta}{\gamma_1\eta}.$$

Note that $f_1(s_0) = 0$.

$(\mathcal{C}_{22})$ $\gamma_{n+1} < \gamma_n$ and there exists α such that

$$\frac{\delta_0\eta}{1-\gamma_0\eta} \leq \alpha \leq \min\{s^\star, s_0\} < 1.$$

See $(\mathcal{C}_2)$ for the existence and uniqueness of zeros s_n.

$(\mathcal{C}_{33})$ $\gamma_{n+1} \leq \gamma_n$ and there exists α such that

$$\frac{\delta_0\eta}{1-\gamma_0\eta} \leq \alpha \leq \min\{r^\star, s_0\} < 1,$$

where,

$$r^\star = \inf r_n.$$

With the notation of Lemma 3.1, we have the following result.

Corollary 3.2. *Assume together with (3.3) that any of conditions $(\mathcal{C}_i)$, $(\mathcal{C}_{ii})$ $(i = 1, 2, 3)$ hold. Then, the conclusions of Lemma 3.1 hold.*

Proof. We note that

$$g_n(s) = \overline{g}_n(s) + (\gamma_{n+1} - \gamma_n)(1 + s + \cdots + s^n) \tag{3.21}$$

and r_n are the unique positive roots of polynomials $\overline{g}_n$.

Case $(\mathcal{C}_1)$. We have $g_n(\alpha) \geq 0$ since $\alpha \geq s^\star$ and for $s \in (0, 1)$

$$\begin{aligned} f_\infty(s) &= \lim_{n\longrightarrow\infty} (\delta_n s^{n-1}\eta + \gamma_n(1 + s + \cdots + s^n)\eta - 1) \\ &\leq b \lim_{n\longrightarrow\infty} s^{n-1}\eta + a \lim_{n\longrightarrow\infty} \frac{1 - s^{n+1}}{1 - s}\eta - 1 \\ &= \frac{a\eta}{1 - s} - 1. \end{aligned}$$

In particular, $f_\infty(1 - a\eta) = 0$, so by the choice of α, we conclude $f_\infty(\alpha) \leq 0$.

Case $(\mathcal{C}_2)$. This part follows from $(\mathcal{C}_1)$ and the remark after $(\mathcal{C}_1)$.

Case $(\mathcal{C}_3)$. It follows from (3.21), $\gamma_{n+1} \leq \gamma_n$ and the choice of α that (3.5) holds. Inequality (3.8) holds in all case $(\mathcal{C}_{ii})$ $(i = 1, 2, 3)$, since $\alpha \leq s_0$. Moreover, (3.9) holds by $\alpha \leq s^\star$.

That completes the proof of Corollary 3.2. □

The following is a special case of $(\mathcal{C}_3)$.

Corollary 3.3 ([7]). *Assume there exist constants $L_0 \geq 0$, $L \geq 0$ and $\eta \geq 0$, with $L_0 \leq L$, such that:*

$$h_{AH} = \bar{L}\eta \begin{cases} \leq \dfrac{1}{2} & \text{if } L_0 \neq 0 \\ < \dfrac{1}{2} & \text{if } L_0 = 0, \end{cases} \tag{3.22}$$

where,

$$\bar{L} = \frac{1}{8}\left(L + 4L_0 + \sqrt{L^2 + 8L_0 L}\right).$$

Then, sequence $\{t_k\}$ $(k \geq 0)$ given by

$$t_0 = 0, \qquad t_1 = \eta, \qquad t_{k+1} = t_k + \frac{L(t_k - t_{k-1})^2}{2(1 - L_0 t_k)} \quad (k \geq 1),$$

is nondecreasing, bounded above by $t^{\star\star}$ and converges to its unique least upper bound $t^\star \in [0, t^{\star\star}]$, where

$$\begin{aligned} t^{\star\star} &= \frac{2\eta}{2 - \delta}, \\ \delta &= \frac{4L}{L + \sqrt{L^2 + 8L_0 L}} < 2 \quad \text{for } L_0 \neq 0. \end{aligned}$$

Moreover the following estimates hold:

$$\begin{aligned} & L_0 t^\star \leq 1, \\ & 0 \leq t_{k+1} - t_k \leq \frac{\delta}{2}(t_k - t_{k-1}) \leq \cdots \leq \left(\frac{\delta}{2}\right)^k \eta, \quad (k \geq 1), \\ & t_{k+1} - t_k \leq \left(\frac{\delta}{2}\right)^k (2h_{AH})^{2^k - 1}\eta, \quad (k \geq 0), \\ & 0 \leq t^\star - t_k \leq \left(\frac{\delta}{2}\right)^k \frac{(2h_{AH})^{2^k - 1}\eta}{1 - (2h_{AH})^{2^k}}, \quad (2h_{AH} < 1),\ (k \geq 0). \end{aligned}$$

Proof. Set $\gamma_n = a$ and $\delta_n = b$ in $(\mathcal{C}_3)$. The proof for the error bounds can be found in [7]. That completes the proof of Corollary 3.3. □

Remark 3.4. (1) If $a = b$, we obtain the famous for its simplicity and clarity Newton–Kantorovich hypothesis for solving nonlinear equations

$$h_K = a\eta \leq \frac{1}{2}. \tag{3.23}$$

As already noted in [7,9,10]

$$h_K \leq \frac{1}{2} \Longrightarrow h_{AH} \leq \frac{1}{2}$$

but not necessarily vice versa unless if $a = b$.
We also have $a \leq b$ holds in general and $\frac{a}{b}$ can be arbitrarily small. We also get

$$\frac{h_{AH}}{h_K} \longrightarrow \frac{1}{4} \quad \text{as } \frac{a}{b} \longrightarrow 0.$$

That is our approach at most quadruples the application of Newton's method. Concerning the error bound, ours are tighter for $a < b$, since $2h_{AH} < 2h_K$. Numerical examples where $a < b$ can be found in [5–10] and in Section 5.

(2) Scalar sequences $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are functions of t_n (see Section 5, Application 5.1). Then, in the case of $r^\star$, we shall choose an upper bound $\overline{r}^\star$ (or a lower bound $\overline{s}^\star$ in case of $s^\star$) independent of n to replace t_n with such as η or $\frac{\eta}{1-\alpha}$.

4. Semilocal convergence analysis

We provide the main semilocal convergence result of (NTM).

Theorem 4.1. *Let $F\colon \mathcal{D} \subseteq X \longrightarrow \mathcal{Y}$ be a Fréchet-differentiable operator. Let $G : \mathcal{D} \longrightarrow \mathcal{Y}$ be a continuous operator and let $A(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be an approximation of $F'(x)$ $(x \in \mathcal{D})$. Assume that there exist $x_0 \in \mathcal{D}$, $\eta > 0$, a bounded inverse $A(x_0)^{-1}$ of $A(x_0)$, functionals $v : \mathcal{D}^2 \longrightarrow [0, \infty)$, $w : \mathcal{D} \longrightarrow [0, 1)$, $z : \mathcal{D} \longrightarrow [0, \infty)$ and $p : \mathcal{D}^2 \longrightarrow [0, \infty)$ such that for all $x, y \in \mathcal{D}$:*

$$\|A(x_0)^{-1}(F(y) - F(x) - F'(x)(y - x))\| \leq v(x, y)\|y - x\|, \tag{4.1}$$

$$\|A(x_0)^{-1}(A(x) - A(x_0))\| \leq w(x), \tag{4.2}$$

$$\|A(x_0)^{-1}(F'(x) - A(x))\| \leq z(x), \tag{4.3}$$

$$\|A(x_0)^{-1}(G(x) - G(y))\| \leq p(x, y)\|x - y\|, \tag{4.4}$$

$$\eta \geq \|A(x_0)^{-1}(F(x_0) + G(x_0))\|; \tag{4.5}$$

there exists α such that

$$0 < \alpha_n = \frac{\beta_n}{1 - \lambda_n} \leq \alpha < 1 \quad (n \geq 0) \tag{4.6}$$

and

$$\overline{U}(x_0, t^\star) = \{x \in \mathcal{X} : \|x - x_0\| \leq t^\star\} \subseteq \mathcal{D}, \tag{4.7}$$

where,

$$\lambda_n = w(x_{n+1}) \quad \text{and} \quad \beta_n = v(x_n, x_{n+1}) + p(x_n, x_{n+1}) + z(x_n).$$

Then

(a) *Scalar sequence $\{t_n\}$ generated by (1.5) is non-decreasing and convergent to $t^\star$, so that estimates (1.7)–(1.10) hold.*

(b) *Sequence $\{x_n\}$ $(n \geq 0)$ generated by (NTM) is well defined, remains in $\overline{U}(x_0, t^\star)$ for all $n \geq 0$ and converges to a zero $x^\star \in \overline{U}(x_0, t^\star)$ of Eq. (1.1).*
Moreover, the following estimates bounds hold for all $n \geq 1$:

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \leq \alpha^n \eta \tag{4.8}$$

and

$$\|x_n - x^\star\| \leq t^\star - t_n \leq \frac{\alpha^n \eta}{1 - \alpha}. \tag{4.9}$$

Furthermore, if there exists $\alpha^\star$ such that

$$0 < \alpha_n^\star = \frac{\beta_n^\star}{1 - \lambda_n} \leq \alpha^\star < 1 \quad (n \geq 0), \tag{4.10}$$

then, the limit point $x^\star$ is unique zero of (1.1) in $U(x_0, t^\star)$, where

$$\beta_n^\star = v(x_n, x^\star) + p(x_n, x^\star) + z(x_n).$$

Proof.

(a) This part can be found in the introduction of this study.

(b) We shall show using induction that sequence $\{x_k\}$ is well defined,

$$\|x_{k+1} - x_k\| \le t_{k+1} - t_k \tag{4.11}$$

and

$$\overline{U}(x_{k+1}, t^\star - t_{k+1}) \subseteq \overline{U}(x_k, t^\star - t_k). \tag{4.12}$$

Let $q \in \overline{U}(x_1, t^\star - t_1)$. Then, estimate

$$\begin{aligned} \|q - x_0\| &\le \|q - x_1\| + \|x_1 - x_0\| \\ &\le t^\star - t_1 + t_1 - t_0 = t^\star - t_0, \end{aligned} \tag{4.13}$$

implies $q \in \overline{U}(x_0, t^\star - t_0)$. We also have

$$\|x_1 - x_0\| = \|A_0^{-1}(F(x_0) + G(x_0))\| \le \eta = t_1 - t_0, \tag{4.14}$$

which together with (4.13) show (4.11) and (4.12) for $k = 0$.

Let us assume $\{x_k\}$ is well defined for all $k \le n$. Using the induction hypotheses we get

$$\|x_k - x_0\| \le \sum_{i=1}^{k} \|x_i - x_{i-1}\| \le \sum_{i=1}^{k} (t_i - t_{i-1}) = t_k - t_0 = t_k < t^\star.$$

That is $x_k \in \overline{U}(x_0, t^\star)$.

Using (1.2), we obtain the approximation

$$\begin{aligned} F(x_{k+1}) + G(x_{k+1}) &= F(x_{k+1}) + G(x_{k+1}) - F(x_k) - G(x_k) - A(x_k)(x_{k+1} - x_k) \\ &= (F(x_{k+1}) - F(x_k) - F'(x_k)(x_{k+1} - x_k)) + (F'(x_k) - A(x_k))(x_{k+1} - x_k) \\ &\quad + (G(x_{k+1}) - G(x_k)). \end{aligned} \tag{4.15}$$

It then follows from (4.1), (4.3), (4.4) and (4.15), the definition of β_k and the induction hypotheses that

$$\begin{aligned} \|A(x_0)^{-1}(F(x_{k+1}) + G(x_{k+1}))\| &\le (v(x_k, x_{k+1}) + p(x_k, x_{k+1}) + z(x_{k+1}))\|x_{k+1} - x_k\| \\ &\le \beta_k(t_{k+1} - t_k). \end{aligned} \tag{4.16}$$

Using (4.2) for $x = x_{k+1}$ and the Banach lemma on invertible operators [5,19], we have $A(x_{k+1})^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ so that

$$\|A(x_{k+1})^{-1}A(x_0)\| \le \frac{1}{1 - w(x_{k+1})} = \frac{1}{1 - \lambda_k}. \tag{4.17}$$

Then, by (1.2), (1.5), (4.16) and (4.17), we get

$$\begin{aligned} \|x_{k+2} - x_{k+1}\| &\le \|A(x_{k+1})^{-1}A(x_0)\| \, \|A(x_0)^{-1}(F(x_{k+1}) + G(x_{k+1}))\| \\ &\le \frac{\beta_k(t_{k+1} - t_k)}{1 - \lambda_k} = \alpha_k(t_{k+1} - t_k) = t_{k+2} - t_{k+1}, \end{aligned} \tag{4.18}$$

which completes the induction for (4.11).

Let $q_0 \in \overline{U}(x_{k+1}, t^\star - t_{k+1})$, then we get

$$\begin{aligned} \|q_0 - x_k\| &\le \|q_0 - x_{k+1}\| + \|x_{k+1} - x_k\| \\ &\le t^\star - t_{k+1} + t_{k+1} - t_k = t^\star - t_k, \end{aligned} \tag{4.19}$$

which shows (4.12) for all k.

The induction for (4.11) and (4.12) is completed. In view of (4.11) and (4.12), sequence $\{x_k\}$ $(k \ge 0)$ is Cauchy (since $\{t_n\}$ is Cauchy) in a Banach space $\mathcal{X}$ and as such it converges to some $x^\star \in \overline{U}(x_0, t^\star)$ (since $\overline{U}(x_0, t^\star)$ is a closed set). By letting $k \longrightarrow \infty$ in (4.16), we get $F(x^\star) + G(x^\star) = 0$ (since $A(x_0)^{-1}$, β_k are bounded and $\{t_k\}$ is a Cauchy sequence). Estimate (4.9) follows from (4.11) by using standard majorization techniques [5,19].

Finally to show the uniqueness part, let $y^\star$ be a zero of $F + G$ in $\overline{U}(x_0, t^\star)$. Using (1.2), we get the approximation

$$\begin{aligned} x_{k+1} - y^\star &= A(x_0)^{-1}(F(y^\star) - F(x_k) - F'(x_k)(y^\star - x_k)) + A(x_0)^{-1}(F'(x_k) - A(x_k))(y^\star - x_k) \\ &\quad + A(x_0)^{-1}(G(y^\star) - G(x_k)). \end{aligned} \tag{4.20}$$

Using (1.4), (4.1)–(4.4), (4.17) and (4.20), we get

$$\begin{aligned} \|x_{k+1} - y^\star\| &\le \frac{\beta_n^\star}{1 - \lambda_n}\|x_k - y^\star\| \\ &= \alpha_n^\star\|x_k - y^\star\| \le \alpha^\star\|x_k - y^\star\| < \|x_k - y^\star\|, \end{aligned} \tag{4.21}$$

from which we deduce $x^\star = y^\star$.

That completes the proof of Theorem 4.1. □

Note that $t^{\star\star}$ given in closed form by (1.7) can replace $t^\star$ in Theorem 4.1.

5. Special cases and applications

In this section we show how to choose sequences and functions introduced in the previous sections for some interesting cases.

Application 5.1. *Newton's method (1.3). Let $G(x) = 0$ and $A(x) = F'(x)$ $(x \in \mathcal{D})$.*

(i) *Assume the Lipschitz conditions [5,8]*

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \le b\|x - y\|,$$
$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \le a\|x - x_0\| \quad \text{for all } x, y \in U(x_0, t^\star) \subseteq \mathcal{D}.$$

Choose:

$$\beta_n = b, \qquad \gamma_n = a \quad (n \ge 0),$$
$$v(x, y) = \left\| \int_0^1 (F'(x + t(y - x)) - F'(x))\, dt \right\|, \quad z(x) = 0,$$
$$w(x) = a\|x - x_0\|, \quad p(x, y) = 0,$$

for all $x, y \in \mathcal{D}$.
Then delicate conditions (4.6) and (4.10) reduce to (3.22) and $at^\star \le 1$ (or $(a + b)t^\star < 1$), respectively.
The advantages of this approach over the one given in the Newton–Kantorovich theorem for solving nonlinear equations [5,8] have already been given in Corollary 3.2 with $(\mathcal{C}_3)$, Corollary 3.3 and Remark 3.4.
According to Corollary 3.2 with $(\mathcal{C}_3)$ and Remark 3.4, our condition (3.22) can be weakened if we can find upper bounds on γ_n and δ_n tighter than a and b, respectively. Below we present such cases.

(ii) *Assume [1,5,8]*

$$\|F'(x_0)^{-1}F(x_0)\| \le \eta,$$
$$\|F'(x_0)^{-1}F''(x_0)\| \le b_0,$$
$$\|F'(x_0)^{-1}(F''(x) - F''(x_0))\| \le b_1\|x - x_0\|$$

and

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \le a\|x - x_0\| \quad \text{for all } x \in U(x_0, r) \subseteq \mathcal{D}.$$

Define polynomial p on $[0, \infty)$ by

$$p(s) = \frac{b_1}{6}s^3 + \frac{b_0}{2}s^2 - s + \eta.$$

Note that we must have $p(r) = 0$.
In view of the estimate

$$F(y) - F(x) - F'(x)(y - x) = \int_0^1 F''(x + t(y - x))(1 - t)(y - x)^2\, dt,$$

we can choose

$$v(x, y) = \left\| \int_0^1 F''(x + t(y - x))(1 - t)(y - x)\, dt \right\|, \qquad w(x) = a\|x - x_0\|,$$
$$z(x) = 0, \qquad p(x, y) = 0, \qquad \gamma_n = a, \qquad \delta_n = \frac{1}{2}\left(\frac{b_1}{3}(t_{n+1} - t_n) + b_0\right),$$
$$b_2 = \frac{1}{2}\left(\frac{b_1\eta}{3(1 - \alpha)} + b_0\right), \qquad \overline{b}_2 = \frac{1}{2}\left(\frac{b_1\eta}{3} + b_0\right)$$

and

$$r^\star \ge \overline{r}^\star = \frac{2b_2}{\overline{b}_2 + \sqrt{\overline{b}_2^2 + 4a\overline{b}_2}},$$

for all $x, y \in \mathcal{D}$.
If $b_2 < b$ (used in (i) above), then hypothesis (3.22) with b_2 replacing b is weaker than in case (i).
The sufficient convergence condition already in the literature [1,5,8] is

$$\eta \le \frac{(b_0^2 + 2b_1)^{3/2} - 3b_0b_1 - b_0^3}{3b_1^2},$$

which is different from (3.22) with b being b_2 or not.

(iii) *Assume* [1,5,8]

$$\|F'(x_0)^{-1}F(x_0)\| \le \eta,$$
$$\|F'(x_0)^{-1}F''(x)\| \le p_1''(\|x-x_0\|) = \frac{2\gamma}{(1-\gamma\|x-x_0\|)^3}$$

and

$$\|F'(x_0)^{-1}(F'(x)-F'(x_0))\| \le a\|x-x_0\| \quad \text{for all } x \in U\left(x_0, \frac{1}{\gamma}\left(1-\frac{1}{\sqrt{2}}\right)\right) \subseteq \mathcal{D},$$

where,

$$p_1(s) = \frac{\gamma s^2}{1-\gamma s} - s + \eta.$$

We can choose

$$v(x,y) = \left\| \int_0^1 F''(x+t(y-x))(1-t)(y-x)\,dt \right\|, \qquad w(x) = a\|x-x_0\|,$$
$$z(x) = 0, \qquad p(x,y) = 0, \qquad \gamma_n = a, \qquad \delta_n = \int_0^1 p_1''(t_n + t(t_{n+1}-t_n))(1-t)(t_{n+1}-t_n)\,dt$$

and

$$b_3 = \frac{1}{2}p_1''\left(\frac{\eta}{1-\alpha}\right).$$

If $b_3 < b$ (b as used in (i) *above), then hypothesis* (3.22) *with b_3 replacing b in weaker than in case* (i).
The sufficient convergence condition [1,5,8] *is*

$$\eta \le \frac{1}{\gamma}(3-2\sqrt{2}).$$

The three cases in Application 5.1 *constitute an incomplete list of possibilities. Note that in practice, we will test these conditions to see which one is verified (if any). If more than one set of conditions is verified, then we will use the one providing the tighter error bounds on the distances $\|x_{n+1}-x_n\|$, $\|x_n-x^\star\|$ and the more precise information on the location of the solution $x^\star$.*

Application 5.2. *Newton-type method* (1.2).
Assume the following conditions [5,8]

$$\|A(x_0)^{-1}(F(x_0)+G(x_0))\| \le \eta,$$
$$\|A'(x_0)^{-1}(F'(x)-F'(y))\| \le c_1\|x-y\|,$$
$$\|A'(x_0)^{-1}(F'(x)-A(x))\| \le c_2\|x-x_0\| + c_3,$$
$$\|A'(x_0)^{-1}(A(x)-A(x_0))\| \le c_4\|x-x_0\| + c_5,$$
$$\|A'(x_0)^{-1}(G(x)-G(y))\| \le c_6\|x-y\|,$$

for all $x, y \in U(x_0, r) \subseteq \mathcal{D}$, where,

$$p_2(s) = \frac{\sigma}{2}s^2 - (1-\sigma_0)s + \eta, \quad \sigma_0 = c_3 + c_5,$$
$$\sigma = \max\{c_1, c_2 + c_4\} \quad \text{and} \quad p_2(r) = 0.$$

We can choose for $x, y \in \mathcal{D}$:

$$v(x,y) = \left\| \int_0^1 (F'(x+t(y-x)) - F'(x))\,dt\,(x-y) \right\|,$$
$$w(x) = c_4\|x-x_0\| + c_5, \qquad p(x,y) = c_6, \qquad z(x) = c_2\|x-x_0\| + c_3,$$

constant α (used in (4.6)*) by*

$$\alpha = \frac{c_7}{c_8},$$

where,

$$c_7 = \frac{c_4\eta}{1-\alpha} \quad \text{and} \quad c_8 = 1 - \left(\frac{1}{2}\frac{c_1\eta}{1-\alpha} + \frac{c_2\eta}{1-\alpha} + c_3 + c_6\right).$$

The hypothesis in [5,8]

$$\eta \leq \frac{1}{2\sigma}(1-\sigma_0)^2.$$

More general than the above choices of functions v, w and z can be found in [2].

We present now some numerical examples. Other applications and examples are also found in [2–8].

Example 5.3 (*[5,8]*). Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$, be equipped with the max-norm, $x_0 = (1, 1)^T$, $U_0 = \{x : \|x - x_0\| \leq 1 - \varrho\}$, $\varrho \in \left[0, \frac{1}{2}\right)$ and define function F on U_0 by

$$F(x) = (\xi_1^3 - \varrho, \xi_2^3 - \varrho)^T, \qquad x = (\xi_1, \xi_2)^T. \tag{5.1}$$

The Fréchet-derivative of operator F is given by

$$F'(x) = \begin{bmatrix} 3\xi_1^2 & 0 \\ 0 & 3\xi_2^2 \end{bmatrix}.$$

Using Application 5.1 and hypotheses of Theorem 4.1, we get:

$$\eta = \frac{1}{3}(1-\varrho), \qquad a = 3 - \varrho \quad \text{and} \quad b = 2\,(2-\varrho).$$

The Kantorovich condition (3.23) is violated, since

$$2h_K = \frac{4}{3}(1-\varrho)(2-\varrho) > 1 \quad \text{for all } \varrho \in \left[0, \frac{1}{2}\right).$$

Hence, there is no guarantee that Newton's method (1.3) converges to $x^\star = (\sqrt[3]{\varrho}, \sqrt[3]{\varrho})^T$, starting at x_0.

However, our condition (3.22) is true for all $\varrho \in I = \left[0.450339002, \frac{1}{2}\right)$. Hence, the conclusions of our Theorem 4.1 can apply to solve Eq. (5.1) for all $\varrho \in I$.

Example 5.4. Define the scalar function F by $F(x) = d_0 x + d_1 + d_2 \sin e^{d_3 x}$, $x_0 = 0$, where d_i, $i = 0, 1, 2, 3$ are given parameters. Then it can easily be seen that for d_3 large and d_2 sufficiently small, $\frac{a}{b}$ can be arbitrarily large.

Example 5.5 (*[5,8]*). Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$ be the space of real-valued continuous functions defined on the interval $[0, 1]$ with norm

$$\|x\| = \max_{0 \leq s \leq 1} |x(s)|.$$

Let $\theta \in [0, 1]$ be a given parameter. Consider the "Cubic" integral equation

$$u(s) = u^3(s) + \lambda u(s) \int_0^1 q(s,t) u(t)\, dt + y(s) - \theta. \tag{5.2}$$

Here the kernel $q(s, t)$ is a continuous function of two variables defined on $[0, 1] \times [0, 1]$; the parameter λ is a real number called the "albedo" for scattering; $y(s)$ is a given continuous function defined on $[0, 1]$ and $x(s)$ is the unknown function sought in $\mathcal{C}[0, 1]$. Equations of the form (5.2) arise in the kinetic theory of gasses [5]. For simplicity, we choose $u_0(s) = y(s) = 1$ and $q(s, t) = \frac{s}{s+t}$, for all $s \in [0, 1]$ and $t \in [0, 1]$, with $s + t \neq 0$. If we let $\mathcal{D} = U(u_0, 1 - \theta)$ and define the operator F on $\mathcal{D}$ by

$$F(x)(s) = x^3(s) - x(s) + \lambda x(s) \int_0^1 q(s,t) x(t)\, dt + y(s) - \theta, \tag{5.3}$$

for all $s \in [0, 1]$, then every zero of F satisfies Eq. (5.2).

We have the estimates

$$\max_{0 \leq s \leq 1} \left| \int_0^1 \frac{s}{s+t}\, dt \right| = \ln 2.$$

Therefore, if we set $\xi = \|F'(u_0)^{-1}\|$, then it follows from Application 5.1 and Theorem 4.1 that

$$\eta = \xi(|\lambda| \ln 2 + 1 - \theta),$$

$$b = 2\xi(|\lambda| \ln 2 + 3(2-\theta)) \quad \text{and} \quad a = \xi(2|\lambda| \ln 2 + 3(3-\theta)).$$

It follows from Application 5.1 and Theorem 4.1 that if our condition (3.22) holds, then problem (5.2) has a unique solution near u_0. This assumption is weaker than the one given before using the Newton–Kantorovich hypothesis (3.23).

Note also that $a < b$ for all $\theta \in [0, 1]$.

Example 5.6. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$, equipped with the max-norm. Consider the following nonlinear boundary value problem [5]

$$\begin{cases} u'' = -u^3 - \gamma u^2 \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 \mathcal{Q}(s, t)(u^3(t) + \gamma u^2(t))\, dt \tag{5.4}$$

where $\mathcal{Q}$ is the Green function:

$$\mathcal{Q}(s, t) = \begin{cases} t(1-s), & t \le s \\ s(1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0 \le s \le 1} \int_0^1 |\mathcal{Q}(s, t)|\, dt = \frac{1}{8}.$$

Then problem (5.4) is in the form (1.1), where, $F : \mathcal{D} \longrightarrow \mathcal{Y}$ is defined as

$$[F(x)](s) = x(s) - s - \int_0^1 \mathcal{Q}(s, t)(x^3(t) + \gamma x^2(t))\, dt.$$

If we set $u_0(s) = s$ and $\mathcal{D} = U(u_0, R)$, then since $\|u_0\| = 1$, it is easy to verify that $U(u_0, R) \subset U(0, R+1)$. If $2\gamma < 5$, then, the operator F' satisfies conditions of Application 5.1, with

$$\eta = \frac{1+\gamma}{5-2\gamma}, \qquad b = \frac{\gamma + 6R + 3}{4}, \qquad a = \frac{2\gamma + 3R + 6}{8}.$$

Note that $a < b$.

6. Conclusion

In order to approximate a locally unique solution of nonlinear equation in a Banach spaces, we provided two very general majorizing sequences using our new concept of recurrent functions. Combining more general Lipschitz and center-Lipschitz conditions on the Fréchet-derivative of the involved operator, we obtained new semilocal convergence analysis of (NTM). Our results extend the applicability of (NTM) studied in [13,16,19–21,30]. Applications and numerical examples are also provided in this study.

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